

3.6 A Summary of Curve Sketching

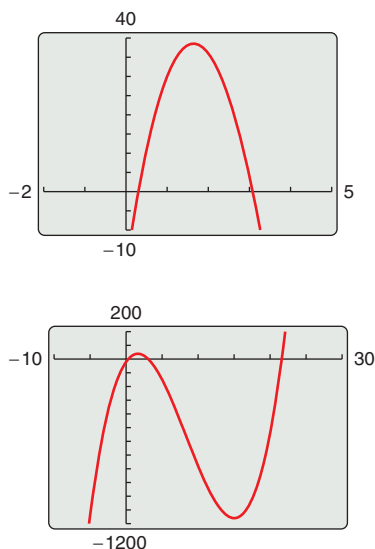
■ Analyze and sketch the graph of a function.

Analyzing the Graph of a Function

It would be difficult to overstate the importance of using graphs in mathematics. Descartes's introduction of analytic geometry contributed significantly to the rapid advances in calculus that began during the mid-seventeenth century. In the words of Lagrange, "As long as algebra and geometry traveled separate paths their advance was slow and their applications limited. But when these two sciences joined company, they drew from each other fresh vitality and thenceforth marched on at a rapid pace toward perfection."

So far, you have studied several concepts that are useful in analyzing the graph of a function.

- x -intercepts and y -intercepts (Section P.1)
- Symmetry (Section P.1)
- Domain and range (Section P.3)
- Continuity (Section 1.4)
- Vertical asymptotes (Section 1.5)
- Differentiability (Section 2.1)
- Relative extrema (Section 3.1)
- Concavity (Section 3.4)
- Points of inflection (Section 3.4)
- Horizontal asymptotes (Section 3.5)
- Infinite limits at infinity (Section 3.5)



Different viewing windows for the graph of $f(x) = x^3 - 25x^2 + 74x - 20$
Figure 3.44

When you are sketching the graph of a function, either by hand or with a graphing utility, remember that normally you cannot show the *entire* graph. The decision as to which part of the graph you choose to show is often crucial. For instance, which of the viewing windows in Figure 3.44 better represents the graph of

$$f(x) = x^3 - 25x^2 + 74x - 20?$$

By seeing both views, it is clear that the second viewing window gives a more complete representation of the graph. But would a third viewing window reveal other interesting portions of the graph? To answer this, you need to use calculus to interpret the first and second derivatives. Here are some guidelines for determining a good viewing window for the graph of a function.

GUIDELINES FOR ANALYZING THE GRAPH OF A FUNCTION

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x -values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use the results to determine relative extrema and points of inflection.

REMARK In these guidelines, note the importance of *algebra* (as well as calculus) for solving the equations

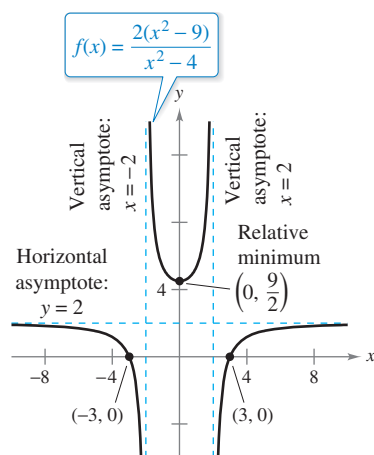
$$f(x) = 0, \quad f'(x) = 0, \quad \text{and} \quad f''(x) = 0.$$

EXAMPLE 1 Sketching the Graph of a Rational Function

Analyze and sketch the graph of

$$f(x) = \frac{2(x^2 - 9)}{x^2 - 4}.$$

Solution

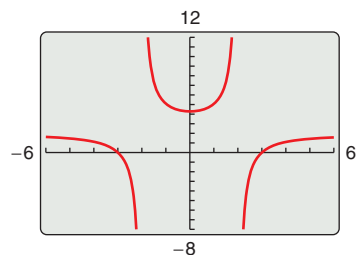


Using calculus, you can be certain that you have determined all characteristics of the graph of f .

Figure 3.45

FOR FURTHER INFORMATION

For more information on the use of technology to graph rational functions, see the article “Graphs of Rational Functions for Computer Assisted Calculus” by Stan Byrd and Terry Walters in *The College Mathematics Journal*. To view this article, go to MathArticles.com.



By not using calculus, you may overlook important characteristics of the graph of g .

Figure 3.46

First derivative: $f'(x) = \frac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \frac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $(0, \frac{9}{2})$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

Critical number: $x = 0$

Possible points of inflection: None

Domain: All real numbers except $x = \pm 2$

Symmetry: With respect to y-axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2, \infty)$

The table shows how the test intervals are used to determine several characteristics of the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < -2$		—	—	Decreasing, concave downward
$x = -2$	Undef.	Undef.	Undef.	Vertical asymptote
$-2 < x < 0$		—	+	Decreasing, concave upward
$x = 0$	$\frac{9}{2}$	0	+	Relative minimum
$0 < x < 2$		+	+	Increasing, concave upward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < \infty$		+	—	Increasing, concave downward

Be sure you understand all of the implications of creating a table such as that shown in Example 1. By using calculus, you can be *sure* that the graph has no relative extrema or points of inflection other than those shown in Figure 3.45.

TECHNOLOGY PITFALL Without using the type of analysis outlined in Example 1, it is easy to obtain an incomplete view of a graph's basic characteristics. For instance, Figure 3.46 shows a view of the graph of

$$g(x) = \frac{2(x^2 - 9)(x - 20)}{(x^2 - 4)(x - 21)}.$$

From this view, it appears that the graph of g is about the same as the graph of f shown in Figure 3.45. The graphs of these two functions, however, differ significantly. Try enlarging the viewing window to see the differences.

EXAMPLE 2 Sketching the Graph of a Rational Function

Analyze and sketch the graph of $f(x) = \frac{x^2 - 2x + 4}{x - 2}$.

Solution

First derivative: $f'(x) = \frac{x(x - 4)}{(x - 2)^2}$

Second derivative: $f''(x) = \frac{8}{(x - 2)^3}$

x -intercepts: None

y -intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim_{x \rightarrow -\infty} f(x) = -\infty, \lim_{x \rightarrow \infty} f(x) = \infty$

Critical numbers: $x = 0, x = 4$

Possible points of inflection: None

Domain: All real numbers except $x = 2$

Test intervals: $(-\infty, 0), (0, 2), (2, 4), (4, \infty)$

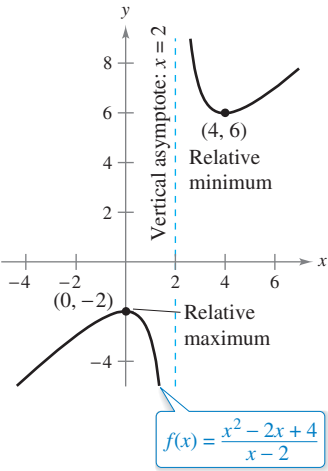
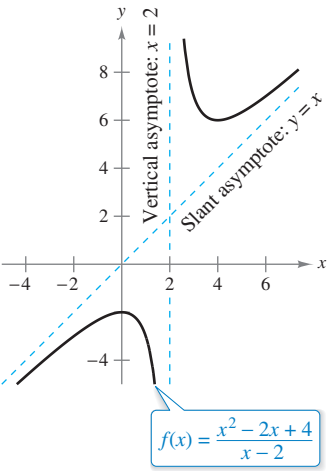


Figure 3.47

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.47.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	−	Increasing, concave downward
$x = 0$	−2	0	−	Relative maximum
$0 < x < 2$		−	−	Decreasing, concave downward
$x = 2$	Undef.	Undef.	Undef.	Vertical asymptote
$2 < x < 4$		−	+	Decreasing, concave upward
$x = 4$	6	0	+	Relative minimum
$4 < x < \infty$		+	+	Increasing, concave upward



A slant asymptote
Figure 3.48

Although the graph of the function in Example 2 has no horizontal asymptote, it does have a slant asymptote. The graph of a rational function (having no common factors and whose denominator is of degree 1 or greater) has a **slant asymptote** when the degree of the numerator exceeds the degree of the denominator by exactly 1. To find the slant asymptote, use long division to rewrite the rational function as the sum of a first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2}$$

Write original equation.

$$= x + \frac{4}{x - 2}$$

Rewrite using long division.

In Figure 3.48, note that the graph of f approaches the slant asymptote $y = x$ as x approaches $-\infty$ or ∞ .

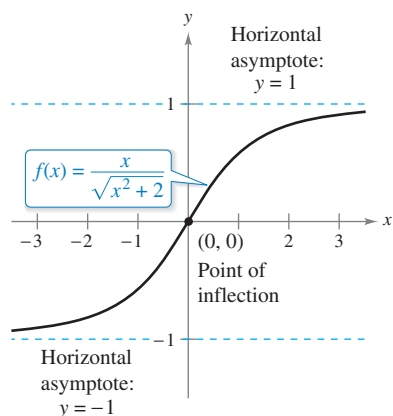
EXAMPLE 3 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \frac{x}{\sqrt{x^2 + 2}}$.

Solution

$$f'(x) = \frac{2}{(x^2 + 2)^{3/2}} \quad \text{Find first derivative.}$$

$$f''(x) = -\frac{6x}{(x^2 + 2)^{5/2}} \quad \text{Find second derivative.}$$

**Figure 3.49**

The graph has only one intercept, $(0, 0)$. It has no vertical asymptotes, but it has two horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left). The function has no critical numbers and one possible point of inflection (at $x = 0$). The domain of the function is all real numbers, and the graph is symmetric with respect to the origin. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.49.

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	+	Increasing, concave upward
$x = 0$	0	$\frac{1}{\sqrt{2}}$	0	Point of inflection
$0 < x < \infty$		+	−	Increasing, concave downward

EXAMPLE 4 Sketching the Graph of a Radical Function

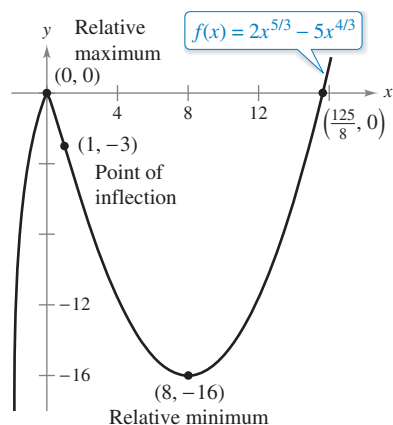
Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \quad \text{Find first derivative.}$$

$$f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}} \quad \text{Find second derivative.}$$

The function has two intercepts: $(0, 0)$ and $(\frac{125}{8}, 0)$. There are no horizontal or vertical asymptotes. The function has two critical numbers ($x = 0$ and $x = 8$) and two possible points of inflection ($x = 0$ and $x = 1$). The domain is all real numbers. The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.50.

**Figure 3.50**

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$-\infty < x < 0$		+	−	Increasing, concave downward
$x = 0$	0	0	Undef.	Relative maximum
$0 < x < 1$		−	−	Decreasing, concave downward
$x = 1$	−3	−	0	Point of inflection
$1 < x < 8$		−	+	Decreasing, concave upward
$x = 8$	−16	0	+	Relative minimum
$8 < x < \infty$		+	+	Increasing, concave upward

EXAMPLE 5 Sketching the Graph of a Polynomial Function

See LarsonCalculus.com for an interactive version of this type of example.

Analyze and sketch the graph of

f(x) = x^4 - 12x^3 + 48x^2 - 64x.

Solution Begin by factoring to obtain

f(x) = x^4 - 12x^3 + 48x^2 - 64x = x(x - 4)^3.

Then, using the factored form of f(x), you can perform the following analysis.

First derivative: f'(x) = 4(x - 1)(x - 4)^2

Second derivative: f''(x) = 12(x - 4)(x - 2)

x-intercepts: (0, 0), (4, 0)

y-intercept: (0, 0)

Vertical asymptotes: None

Horizontal asymptotes: None

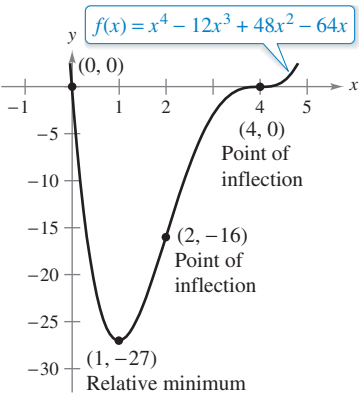
End behavior: lim_{x to -inf} f(x) = inf, lim_{x to inf} f(x) = inf

Critical numbers: x = 1, x = 4

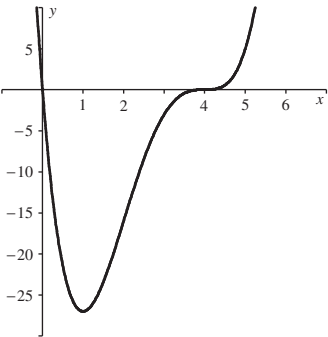
Possible points of inflection: x = 2, x = 4

Domain: All real numbers

Test intervals: (-inf, 1), (1, 2), (2, 4), (4, inf)



(a)



Generated by Maple

(b)

A polynomial function of even degree must have at least one relative extremum.

Figure 3.51

The analysis of the graph of f is shown in the table, and the graph is shown in Figure 3.51(a). Using a computer algebra system such as Maple [see Figure 3.51(b)] can help you verify your analysis.

	f(x)	f'(x)	f''(x)	Characteristic of Graph
$-\infty < x < 1$		-	+	Decreasing, concave upward
$x = 1$	-27	0	+	Relative minimum
$1 < x < 2$		+	+	Increasing, concave upward
$x = 2$	-16	+	0	Point of inflection
$2 < x < 4$		+	-	Increasing, concave downward
$x = 4$	0	0	0	Point of inflection
$4 < x < \infty$		+	+	Increasing, concave upward

The fourth-degree polynomial function in Example 5 has one relative minimum and no relative maxima. In general, a polynomial function of degree n can have at most n - 1 relative extrema, and at most n - 2 points of inflection. Moreover, polynomial functions of even degree must have at least one relative extremum.

Remember from the Leading Coefficient Test described in Section P.3 that the “end behavior” of the graph of a polynomial function is determined by its leading coefficient and its degree. For instance, because the polynomial in Example 5 has a positive leading coefficient, the graph rises to the right. Moreover, because the degree is even, the graph also rises to the left.

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Analyze and sketch the graph of $f(x) = (\cos x)/(1 + \sin x)$.

Solution Because the function has a period of 2π , you can restrict the analysis of the graph to any interval of length 2π . For convenience, choose $(-\pi/2, 3\pi/2)$.

$$\text{First derivative: } f'(x) = -\frac{1}{1 + \sin x}$$

$$\text{Second derivative: } f''(x) = \frac{\cos x}{(1 + \sin x)^2}$$

$$\text{Period: } 2\pi$$

$$\text{x-intercept: } \left(\frac{\pi}{2}, 0\right)$$

$$\text{y-intercept: } (0, 1)$$

$$\text{Vertical asymptotes: } x = -\frac{\pi}{2}, x = \frac{3\pi}{2}$$

See Remark below.

$$\text{Horizontal asymptotes: None}$$

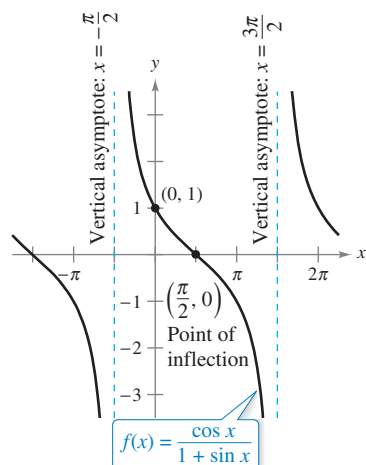
$$\text{Critical numbers: None}$$

$$\text{Possible points of inflection: } x = \frac{\pi}{2}$$

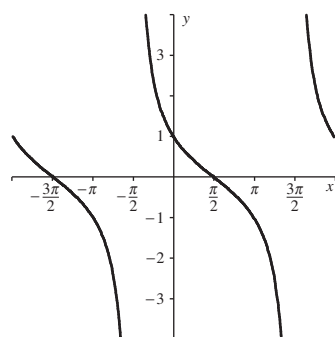
$$\text{Domain: All real numbers except } x = \frac{3 + 4n}{2}\pi$$

$$\text{Test intervals: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

The analysis of the graph of f on the interval $(-\pi/2, 3\pi/2)$ is shown in the table, and the graph is shown in Figure 3.52(a). Compare this with the graph generated by the computer algebra system *Maple* in Figure 3.52(b).



(a)



Generated by Maple

(b)

Figure 3.52

	$f(x)$	$f'(x)$	$f''(x)$	Characteristic of Graph
$x = -\frac{\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote
$-\frac{\pi}{2} < x < \frac{\pi}{2}$		—	+	Decreasing, concave upward
$x = \frac{\pi}{2}$	0	$-\frac{1}{2}$	0	Point of inflection
$\frac{\pi}{2} < x < \frac{3\pi}{2}$		—	—	Decreasing, concave downward
$x = \frac{3\pi}{2}$	Undef.	Undef.	Undef.	Vertical asymptote

REMARK By substituting $-\pi/2$ or $3\pi/2$ into the function, you obtain the form $0/0$. This is called an indeterminate form, which you will study in Section 8.7. To determine that the function has vertical asymptotes at these two values, rewrite f as

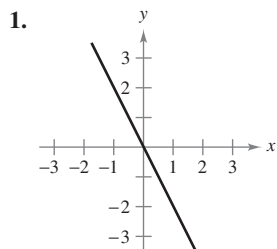
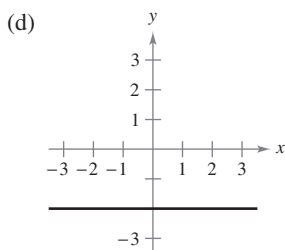
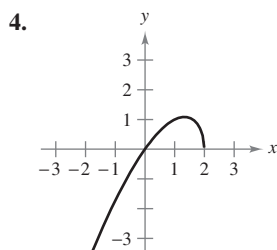
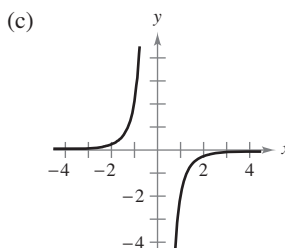
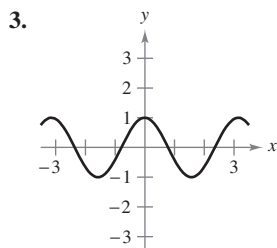
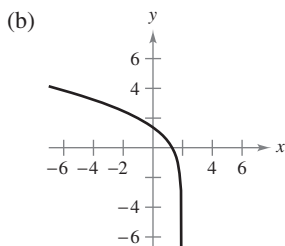
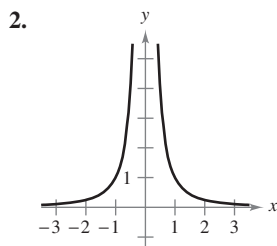
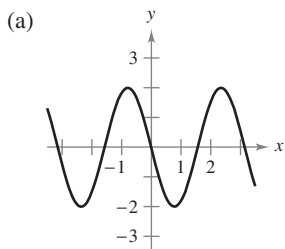
$$f(x) = \frac{\cos x}{1 + \sin x} = \frac{(\cos x)(1 - \sin x)}{(1 + \sin x)(1 - \sin x)} = \frac{(\cos x)(1 - \sin x)}{\cos^2 x} = \frac{1 - \sin x}{\cos x}.$$

In this form, it is clear that the graph of f has vertical asymptotes at $x = -\pi/2$ and $3\pi/2$.

3.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the graph of f in the left column with that of its derivative in the right column.

Graph of f Graph of f' 

Analyzing the Graph of a Function In Exercises 5–24, analyze and sketch a graph of the function. Label any intercepts, relative extrema, points of inflection, and asymptotes. Use a graphing utility to verify your results.

5. $y = \frac{1}{x-2} - 3$

6. $y = \frac{x}{x^2 + 1}$

7. $y = \frac{x^2}{x^2 + 3}$

8. $y = \frac{x^2 + 1}{x^2 - 4}$

9. $y = \frac{3x}{x^2 - 1}$

10. $f(x) = \frac{x-3}{x}$

11. $f(x) = x + \frac{32}{x^2}$

12. $f(x) = \frac{x^3}{x^2 - 9}$

13. $y = \frac{x^2 - 6x + 12}{x - 4}$

14. $y = \frac{-x^2 - 4x - 7}{x + 3}$

15. $y = x\sqrt{4-x}$

16. $g(x) = x\sqrt{9-x^2}$

17. $y = 3x^{2/3} - 2x$

18. $y = (x+1)^2 - 3(x+1)^{2/3}$

19. $y = 2 - x - x^3$

20. $y = -\frac{1}{3}(x^3 - 3x + 2)$

21. $y = 3x^4 + 4x^3$

22. $y = -2x^4 + 3x^2$

23. $y = x^5 - 5x$

24. $y = (x-1)^5$



Analyzing the Graph of a Function Using Technology

In Exercises 25–34, use a computer algebra system to analyze and graph the function. Identify any relative extrema, points of inflection, and asymptotes.

25. $f(x) = \frac{20x}{x^2 + 1} - \frac{1}{x}$

26. $f(x) = x + \frac{4}{x^2 + 1}$

27. $f(x) = \frac{-2x}{\sqrt{x^2 + 7}}$

28. $f(x) = \frac{4x}{\sqrt{x^2 + 15}}$

29. $f(x) = 2x - 4 \sin x, \quad 0 \leq x \leq 2\pi$

30. $f(x) = -x + 2 \cos x, \quad 0 \leq x \leq 2\pi$

31. $y = \cos x - \frac{1}{4} \cos 2x, \quad 0 \leq x \leq 2\pi$

32. $y = 2x - \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

33. $y = 2(\csc x + \sec x), \quad 0 < x < \frac{\pi}{2}$

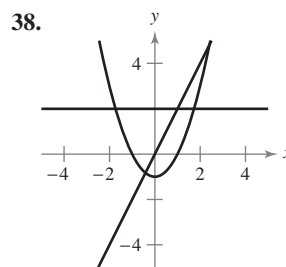
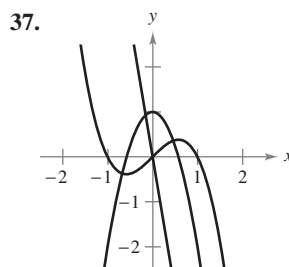
34. $g(x) = x \cot x, \quad -2\pi < x < 2\pi$

WRITING ABOUT CONCEPTS

35. Using a Derivative Let $f'(t) < 0$ for all t in the interval $(2, 8)$. Explain why $f(3) > f(5)$.

36. Using a Derivative Let $f(0) = 3$ and $2 \leq f'(x) \leq 4$ for all x in the interval $[-5, 5]$. Determine the greatest and least possible values of $f(2)$.

Identifying Graphs In Exercises 37 and 38, the graphs of f , f' , and f'' are shown on the same set of coordinate axes. Which is which? Explain your reasoning. To print an enlarged copy of the graph, go to MathGraphs.com.



WRITING ABOUT CONCEPTS (continued)

Horizontal and Vertical Asymptotes In Exercises 39–42, use a graphing utility to graph the function. Use the graph to determine whether it is possible for the graph of a function to cross its horizontal asymptote. Do you think it is possible for the graph of a function to cross its vertical asymptote? Why or why not?

39. $f(x) = \frac{4(x-1)^2}{x^2 - 4x + 5}$

40. $g(x) = \frac{3x^4 - 5x + 3}{x^4 + 1}$

41. $h(x) = \frac{\sin 2x}{x}$

42. $f(x) = \frac{\cos 3x}{4x}$

Examining a Function In Exercises 43 and 44, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function may indicate that there should be one.

43. $h(x) = \frac{6-2x}{3-x}$

44. $g(x) = \frac{x^2 + x - 2}{x - 1}$

Slant Asymptote In Exercises 45–48, use a graphing utility to graph the function and determine the slant asymptote of the graph. Zoom out repeatedly and describe how the graph on the display appears to change. Why does this occur?

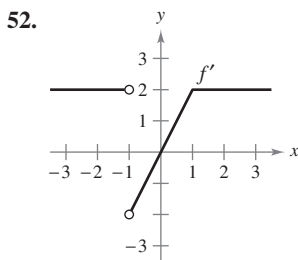
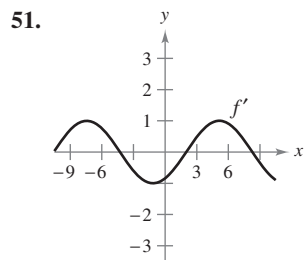
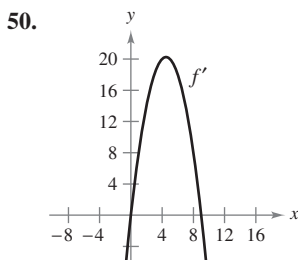
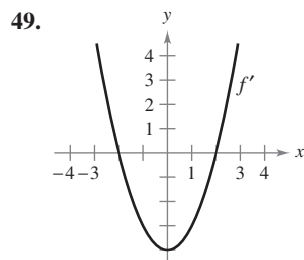
45. $f(x) = -\frac{x^2 - 3x - 1}{x - 2}$

46. $g(x) = \frac{2x^2 - 8x - 15}{x - 5}$

47. $f(x) = \frac{2x^3}{x^2 + 1}$

48. $h(x) = \frac{-x^3 + x^2 + 4}{x^2}$

Graphical Reasoning In Exercises 49–52, use the graph of f' to sketch a graph of f and the graph of f'' . To print an enlarged copy of the graph, go to MathGraphs.com.



(Submitted by Bill Fox, Moberly Area Community College, Moberly, MO)

53. Graphical Reasoning Consider the function

$$f(x) = \frac{\cos^2 \pi x}{\sqrt{x^2 + 1}}, \quad 0 < x < 4.$$

- Use a computer algebra system to graph the function and use the graph to approximate the critical numbers visually.
- Use a computer algebra system to find f' and approximate the critical numbers. Are the results the same as the visual approximation in part (a)? Explain.

54. Graphical Reasoning Consider the function

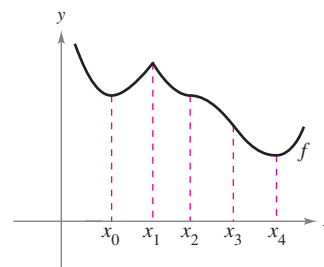
$$f(x) = \tan(\sin \pi x).$$

- Use a graphing utility to graph the function.
- Identify any symmetry of the graph.
- Is the function periodic? If so, what is the period?
- Identify any extrema on $(-1, 1)$.
- Use a graphing utility to determine the concavity of the graph on $(0, 1)$.

Think About It In Exercises 55–58, create a function whose graph has the given characteristics. (There is more than one correct answer.)

- Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 0$
- Vertical asymptote: $x = -5$
Horizontal asymptote: None
- Vertical asymptote: $x = 3$
Slant asymptote: $y = 3x + 2$
- Vertical asymptote: $x = 2$
Slant asymptote: $y = -x$

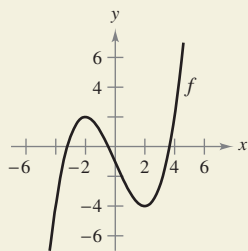
59. Graphical Reasoning Identify the real numbers $x_0, x_1, x_2, x_3,$ and x_4 in the figure such that each of the following is true.



- $f'(x) = 0$
- $f''(x) = 0$
- $f'(x)$ does not exist.
- f has a relative maximum.
- f has a point of inflection.

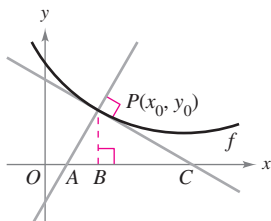


60. HOW DO YOU SEE IT? The graph of f is shown in the figure.



- For which values of x is $f'(x)$ zero? Positive? Negative? What do these values mean?
- For which values of x is $f''(x)$ zero? Positive? Negative? What do these values mean?
- On what open interval is f' an increasing function?
- For which value of x is $f''(x)$ minimum? For this value of x , how does the rate of change of f compare with the rates of change of f for other values of x ? Explain.

61. Investigation Let $P(x_0, y_0)$ be an arbitrary point on the graph of f such that $f'(x_0) \neq 0$, as shown in the figure. Verify each statement.



- The x -intercept of the tangent line is

$$\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right).$$

- The y -intercept of the tangent line is

$$(0, f(x_0) - x_0 f'(x_0)).$$

- The x -intercept of the normal line is

$$(x_0 + f(x_0)f'(x_0), 0).$$

- The y -intercept of the normal line is

$$\left(0, y_0 + \frac{x_0}{f'(x_0)}\right).$$

$$(e) |BC| = \left| \frac{f(x_0)}{f'(x_0)} \right|$$

$$(f) |PC| = \left| \frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)} \right|$$

$$(g) |AB| = |f(x_0)f'(x_0)|$$

$$(h) |AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

62. Investigation Consider the function

$$f(x) = \frac{2x^n}{x^4 + 1}$$

for nonnegative integer values of n .

- Discuss the relationship between the value of n and the symmetry of the graph.
 - For which values of n will the x -axis be the horizontal asymptote?
 - For which value of n will $y = 2$ be the horizontal asymptote?
 - What is the asymptote of the graph when $n = 5$?
- (e) Use a graphing utility to graph f for the indicated values of n in the table. Use the graph to determine the number of extrema M and the number of inflection points N of the graph.

n	0	1	2	3	4	5
M						
N						

63. Graphical Reasoning Consider the function

$$f(x) = \frac{ax}{(x - b)^2}.$$

Determine the effect on the graph of f as a and b are changed. Consider cases where a and b are both positive or both negative, and cases where a and b have opposite signs.

64. Graphical Reasoning Consider the function

$$f(x) = \frac{1}{2}(ax)^2 - ax, \quad a \neq 0.$$

- Determine the changes (if any) in the intercepts, extrema, and concavity of the graph of f when a is varied.
- (b) In the same viewing window, use a graphing utility to graph the function for four different values of a .

Slant Asymptotes In Exercises 65 and 66, the graph of the function has two slant asymptotes. Identify each slant asymptote. Then graph the function and its asymptotes.

65. $y = \sqrt{4 + 16x^2}$

66. $y = \sqrt{x^2 + 6x}$

PUTNAM EXAM CHALLENGE

- 67.** Let $f(x)$ be defined for $a \leq x \leq b$. Assuming appropriate properties of continuity and derivability, prove for $a < x < b$ that

$$\frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a} = \frac{1}{2}f''(\varepsilon),$$

where ε is some number between a and b .

This problem was composed by the Committee on the Putnam Prize Competition.
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